

Fibonacci topological order from quantum nets

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We analyze a model of quantum nets and show it has non-abelian topological order of doubled Fibonacci type. The ground state has the same topological behavior as that of the corresponding string-net model, but our Hamiltonian has less complicated interactions, and the excitations are dynamical, not fixed. This Hamiltonian includes terms acting on the spins around a face, around a vertex, and special “Jones-Wenzl” terms that serve to couple long loops together. We provide strong evidence for a gap by exact diagonalization, completing the list of ingredients necessary for topological order.

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Introduction— The theoretical and experimental search for condensed-matter systems with fractionalized excitations has attracted considerable attention in recent years. One reason is the promise of protection against errors in a topological quantum computer [1, 2]. Another is more fundamental: fractionalization provides a dramatic example of how rich the emergent long-distance behavior of a many-body system can be.

Finding magnetic systems with fractionalized excitations, in particular those with the non-abelian statistics necessary for topological quantum computation, is a difficult but important problem. Major progress resulted from the introduction of the “toric code” or “quantum-double” models, exactly solvable models based on any discrete group [1]. Even more types of fractionalized excitations and their anyonic braiding occur in the “string-net” models [3]. These Hamiltonians are sums of commuting projectors, each of which annihilates the ground state. The terms must be highly fine-tuned so that they commute: for example, the string-net Hamiltonian contains interactions among 12 spins. Since each can be diagonalized individually, the full gapped spectrum of the Hamiltonian is easily computed. The excitations are effectively immobile defects and so are non-dynamical. Even though the solvable models are obviously quite special, the presence of the gap means they exemplify a particular phase, not just a special point. Indeed, rigorous proofs [4, 5] show that the topological order necessary for fractionalized excitations persists when perturbing around the exactly solvable points.

In this paper we focus on a particularly important phase, that with “Fibonacci” topological order. This is the simplest kind of topological order that is not only non-abelian but universal; i.e. any unitary operator on the Hilbert space can be approximated to arbitrary precision by braiding the quasiparticles. We find a Hamiltonian with an exact ground state that is simpler than that of the exactly solvable string-net Fibonacci model [3], and which further illuminates the origin of this resulting topological order. We expand upon Ref. 6 by including terms in the Hamiltonian with the essential property of coupling loops in different winding sectors. By extensive numerical exact diagonalization, we then provide strong evidence that the resulting Hamiltonian is indeed gapped. Coupled with the work of Refs. 6, 7 on the ground state, our work provides

an explicit Hamiltonian with doubled Fibonacci topological order and dynamical excitations.

Quantum loops and nets— The ground state of a system with fractionalized excitations must satisfy special properties. In a magnetic model, excitations are typically attached by a defect line or “string”. The string itself costs no energy, because away from the quasiparticles, it is locally indistinguishable from the ground state itself. The existence of the string does however result in anyonic statistics, because rotating one quasiparticle around another causes the strings to interact, even when the quasiparticles are far apart. This picture is elegantly illustrated in *quantum loop models* [8], where the appropriate topological properties are built in directly. The Hilbert space is spanned by non-intersecting string configurations, and the Hamiltonian includes an energy penalty for each “string end”. Thus the ground state can only be comprised of states containing closed loops. Any state with a string end then corresponds to a quasiparticle excitation, and the defect lines are the strings attaching the quasiparticles. There are thus for example three distinct configurations with four quasiparticles, corresponding to how the strings attach them. Since away from a quasiparticle a string with ends is indeed indistinguishable from a closed loop, these three configurations should have the same energy. This degeneracy opens up the possibility of non-abelian braiding, because braiding can cause a transition between the different configurations.

Unfortunately, subsequent work [9–11] showed that non-abelian anyons do not occur in quantum loop models in their simplest formulation. A way of salvaging the idea is to modify the inner product to one with more desirable topological properties [6]. The end result is that when one transforms a quantum loop model with the appropriate (non-orthogonal) inner product to an orthonormal basis, the excitations are naturally described in terms of *quantum nets*, configurations that allow branching, or vertices with three or more strings present. Thus quite nicely, one ends up at the end with the same sort of Hilbert space as in the string-net models!

The vertex terms in the Hamiltonian— Our Hamiltonian is of “Rokhsar-Kivelson” [12] or “frustration-free” type. It is the sum over (not necessarily commuting) projectors, such that there exists at least one state annihilated by each of the projec-

tors. Such a state is necessarily a ground state, and all ground states must be annihilated by all projectors. Quasiparticle excitations then necessarily involve configurations not annihilated by at least one of these projectors; the lowest-energy excitations are those dominated by configurations where only a few of the projectors acting on it do not annihilate it.

We study models with a two-state orthonormal quantum system $|0\rangle_l, |1\rangle_l$ at each *link* l of some two-dimensional lattice. The basis elements of the Hilbert space are then the tensor product of the states for each link. A convenient way of picturing each basis state is to draw a line on each link l when the state $|1\rangle_l$ is present, while leaving the link empty for $|0\rangle_l$. This way, each basis state is pictured by some geometric object; the strings are formed from links with $|1\rangle_l$.

A quantum net Hamiltonian includes an energy penalty for “string ends”, to ensure none appear in the ground state [1, 3, 8]. This is done by adding a term for each *vertex* v projecting on to any configuration with a string end, i.e. a configuration where there is one link l touching v with the state $|1\rangle_l$ and the remainder of links having states with $|0\rangle_l$. Let the projector P_l act as $P_l(\cdots \otimes |0\rangle_l \otimes \cdots) = (\cdots \otimes |0\rangle_l \otimes \cdots)$ and $P_l(\cdots \otimes |1\rangle_l \otimes \cdots) = 0$. Thus on the square lattice, with v_1, v_2, v_3, v_4 the links touching v , the operator

$$H_v = (1 - P_{v_1})P_{v_2}P_{v_3}P_{v_4} + (1 - P_{v_2})P_{v_1}P_{v_3}P_{v_4} \\ + (1 - P_{v_3})P_{v_1}P_{v_2}P_{v_4} + (1 - P_{v_4})P_{v_1}P_{v_2}P_{v_3}(1)$$

annihilates any basis state without a string end at v , and gives 1 acting on any basis state with one. Adding $\sum_v H_v$ to a Hamiltonian comprised of sums of projectors therefore forbids string ends in a zero-energy ground state. The zero-energy state $|\Psi\rangle$ annihilated by $\sum_v H_v$ can thus be written as a sum over “nets” N , geometric objects that have no ends:

$$|\Psi\rangle = \sum_N w(N)|N\rangle, \quad (2)$$

Other terms in the Hamiltonian then will determine the values of the weights $w(N) = \langle N|\Psi\rangle$ in the ground state.

A ground state with non-abelian topological order— The weights $w(N)$ of the configurations in the ground state (2) must satisfy certain properties to ensure topological order. One property is that the number of ground states depends on the genus of the surface on which the model is defined. For example, in the toric code [1] and in the triangular-lattice quantum dimer model [13], there is only one ground state when space is topologically a sphere, but there are four when space is a torus. A ground state of the form (2) is very natural for giving this behavior, because geometric objects such as loops and nets can have winding numbers around the cycles of a surface with non-trivial topology. When the local terms in the Hamiltonian do not change the winding numbers, there is a ground state for each winding number.

A gap in the spectrum is also highly desirable. While of course a gap is a property of the Hamiltonian, the ground state of any local Hamiltonian with a gap in two dimensions satisfies an important constraint: all expectation values of *local* operators in such a ground state must decay exponentially as the

operators are moved apart [14]. On the flip side, however, to have deconfined anyons, the expectation value of some *non-local* operators must decay algebraically; geometric objects like loops or nets should have long-range correlations. Exponential decay here would confine anyons, because it would imply a vanishing probability for two would-be fractionalized excitations attached by a string to be far apart. A familiar classical model with analogous behavior is percolation: at its threshold the probability that two points are on the same cluster decays algebraically, while by construction all correlators of local quantities have zero correlation length.

To have abelian topological order, a ground state satisfying these properties is typically sufficient. To have fractionalized excitations with non-abelian statistics, however, much more structure need be present. Obviously, there also must be degeneracies in the excited-state spectrum so that the exchange of quasiparticles can cause a transition to another state. However, this is not enough either [9–11]; the inner product must also have the appropriate topological properties [6].

In Ref. 15, it was proposed that non-abelian topological order occurs when the topological part of ground-state weight $w(N)$ is given by a *chromatic polynomial*. The chromatic polynomial $\chi_{\hat{N}}(Q)$ is easiest to understand by treating the strings in the net N as borders separating countries. The dual graph \hat{N} corresponds to a vertex for each country and an edge connecting each pair of countries sharing a border. For Q integer, $\chi_{\hat{N}}(Q)$ is the number of ways of coloring each country with Q colors such that adjacent countries (i.e. two adjacent vertices on \hat{N}) are colored differently. It is a polynomial in Q that can be evaluated for Q non-integer as well, and by definition is a topological invariant. Any string end results in $\chi_{\hat{N}}(Q) = 0$ because the dual has a vertex attached to itself.

We focus on the simplest example of universal non-Abelian topological order, the “Fibonacci” case $Q = \phi^2 = \phi + 1$, where $\phi = (1 + \sqrt{5})/2$, the golden ratio. The ground-state weight $w(N)$ of any net model with (doubled) Fibonacci order, including the string-net model, must necessarily involve the chromatic polynomial [16]. This topological order and the resulting excitations have been discussed in depth in [16, 17]. The degrees of freedom in the Fibonacci string-net model [3] are a two-state system $|0\rangle_l, |1\rangle_l$ on the links of the honeycomb lattice, and the unnormalized ground state $|\Psi\rangle$ in (2) is summed over all net configurations without ends with [16]

$$w(N) = w_s(N) \equiv \phi^{3t_N/4} \chi_{\hat{N}}(\phi^2), \quad (3)$$

where t_N is the number of trivalent vertices in the net N (i.e. those with all three neighboring links in the state $|1\rangle$). To find a Hamiltonian simpler than that of the string-net model, and which has dynamical excitations, we study a ground state slightly different than (3). Following [6], we take

$$w(N) = \phi^{-L_N/2} \chi_{\hat{N}}(\phi^2), \quad (4)$$

where the “length” L_N of the net N is the number of links it covers, i.e. the number of states $|1\rangle_l$. Since the weights (3,4) are identical in their topological properties, it is natural

to expect that the latter results in the same doubled-Fibonacci topological order as the string-net ground state (3). Numerical and analytical arguments indicate that indeed such a ground state has all the desired properties described above [6, 7].

The face terms in the Hamiltonian— The remainder of this paper is devoted to finding a gapped Hamiltonian annihilating the ground state $|\Psi\rangle$ in (2) with weights (4). An advantage of using the weights (4) is that they obey a “quantum self-duality”: they take the same form when rewritten on the dual lattice [6]. The results in projectors annihilating this ground state involving only the links around a face of the lattice, as opposed to the 12-spin interaction in the string-net model [3].

Seeing this requires making a change of basis from $|0\rangle_l, |1\rangle_l$ on each link to another basis $|\hat{0}\rangle_l, |\hat{1}\rangle_l$ via the unitary matrix $F = F^{-1}$ defined by

$$\begin{pmatrix} |\hat{0}\rangle_l \\ |\hat{1}\rangle_l \end{pmatrix} = \frac{1}{\phi} \begin{pmatrix} 1 & \sqrt{\phi} \\ \sqrt{\phi} & -1 \end{pmatrix} \begin{pmatrix} |0\rangle_l \\ |1\rangle_l \end{pmatrix}. \quad (5)$$

Not coincidentally, this matrix is also the fusion matrix for four Fibonacci anyons. The quantum self-duality stems from interpreting this new basis as describing nets on the *dual* lattice. The links of the dual lattice are in one-to-one correspondence with those of the original lattice, so we are free to define such a net D by drawing a string on the *dual* link when the state $|\hat{1}\rangle_l$ is present, and leaving the dual link empty when $|\hat{0}\rangle_l$ is present. The remarkable fact is that when the ground state $|\Psi\rangle$ is rewritten in the new basis, one obtains [6, 18]

$$|\Psi\rangle = \sum_D \hat{w}(D) |D\rangle \quad (6)$$

$$\hat{w}(D) = \langle D | \Psi \rangle = \alpha \phi^{-L_D/2} \chi_{\hat{D}}(\phi^2), \quad (7)$$

where L_D is the length of the net D , i.e. the number of dual links with $|\hat{1}\rangle_l$, while α is an unimportant constant. Comparing (6,7) with (2,4) shows the weighting of the nets of the dual lattice is completely equivalent to the weighting on the original lattice, hence the quantum self-duality.

In particular, (7) means that any configuration D with string ends has $\hat{w}(D)=0$ and does not appear in the ground state. Thus a projector H_f for vertices on the dual lattice, analogous to H_v on the original lattice, will also annihilate this ground state. The links f_1, f_2, \dots touching a vertex on the dual lattice correspond to links around a *face* on the original lattice. Thus defining the projector $\hat{P}_l = F P_l F$ onto $|\hat{0}\rangle_l$ means that for each original face on the square lattice, the projector

$$H_f = (1 - \hat{P}_{f_1}) \hat{P}_{f_2} \hat{P}_{f_3} \hat{P}_{f_4} + (1 - \hat{P}_{f_2}) \hat{P}_{f_1} \hat{P}_{f_3} \hat{P}_{f_4} \quad (8) \\ + (1 - \hat{P}_{f_3}) \hat{P}_{f_1} \hat{P}_{f_2} \hat{P}_{f_4} + (1 - \hat{P}_{f_4}) \hat{P}_{f_1} \hat{P}_{f_2} \hat{P}_{f_3}$$

annihilates $|\Psi\rangle$. This operator is non-diagonal in the original net basis and so couples different net configurations.

The Hamiltonian $H_{\text{vf}} = \sum_v H_v + \sum_f H_f$ thus annihilates the desired ground state $|\Psi\rangle$, and only has interactions around each vertex and face, so e.g. only has four-spin interactions for the square lattice. The last remaining thing to check is if it has a gap. Our numerical results unfortunately indicate that

there is *not* a gap for this Hamiltonian, see fig. 5. Another (probably related) problem is that H_{vf} does not have the right number (four) of ground states for doubled Fibonacci topological order on the torus. The number grows with the size of the torus, which follows by rewriting $|\Psi\rangle$ in a (non-orthonormal) loop basis; the unwanted ground states correspond to loops wrapping around the cycles of the torus [6].

Coupling wrapping loops via Jones-Wenzl terms— A virtue of writing the ground-state weights in terms of topological quantities like the chromatic polynomial is that one can find many local projectors annihilating it [15]. However, many of these will not change the number of ground states on a torus. However, “Jones-Wenzl” projectors [19] added to the Hamiltonian not only couple the long loops and so remove the unwanted ground states, but can give a gap [8]. For example, the toric code Hamiltonian can be rewritten in the form H_{vf} plus Jones-Wenzl type terms appropriate for $Q=2$ [6]. We here derive these terms for the Fibonacci ground state (4), and provide strong evidence using numerical exact diagonalization that the spectrum now includes a gap.

FIG. 1: The Jones-Wenzl identity for $Q = \phi^2$

The Jones-Wenzl projectors for knot polynomials result in an identity for each chromatic polynomial at $\cos^{-1}(\sqrt{Q}/2)/\pi$ rational [18]. The identity for $Q = \phi^2$ is [20]

$$\chi_{\hat{t}}(\phi^2) = \frac{1}{\phi} \chi_{\hat{E}}(\phi^2) - \frac{1}{\phi^2} \chi_{\hat{I}}(\phi^2) \quad (9)$$

with the three nets t, E, I involved illustrated in fig. 1. This identity is true *locally*, meaning it holds for any portion of a net. Consider the three nets $|t\rangle$, $|E\rangle$, and $|I\rangle$ on the square lattice displayed in fig. 2, which are identical everywhere except on one face. Because the ground state (4) is written in terms of chromatic polynomials, the identity (9) results in a local relation between amplitudes in the ground-state wavefunction:

$$\langle t | \Psi \rangle = \phi^{-3/2} \langle E | \Psi \rangle - \phi^{-5/2} \langle I | \Psi \rangle. \quad (10)$$

The extra factor of $\sqrt{\phi}$ in (10) relative to (9) results from the weight per unit length in (4).

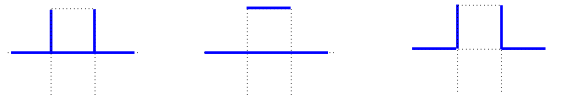


FIG. 2: Three configurations mixed by the Jones-Wenzl projector

We can exploit the relation (10) to find a projector H_j that annihilates the ground state $|\Psi\rangle$ and couples the winding loops. It is of the form $H_j = J_{\text{in}} J_{\text{out}}$, and acts non-diagonally on the four “inside” links around the square in fig. 2, and

diagonally on the remaining four “outside” links. The non-diagonal part of each H_j acting on the inner links is chosen so as to project onto the linear combination $\phi^{5/2}|t\rangle - \phi|E\rangle + |I\rangle$. A Hermitian operator doing this is

$$J_{\text{in}} = \begin{pmatrix} \phi^{5/2} & -\phi & 1 \\ -\phi & \phi^{-1/2} & -\phi^{-3/2} \\ 1 & -\phi^{-3/2} & \phi^{-5/2} \end{pmatrix} \quad (11)$$

acting on $|t\rangle$, $|E\rangle$ and $|I\rangle$ respectively; it annihilates all other configurations on the inner links. The diagonal part J_{out} annihilates any configuration on the outer links other than those illustrated in fig. 3, i.e. labeling the outside links j_1, j_2, j_3 and j_4 from left to right and letting $Q_j = (1 - P_j)$ gives

$$J_{\text{out}} = Q_{j_1}P_{j_2}Q_{j_3}P_{j_4} + Q_{j_2}P_{j_1}Q_{j_3}P_{j_4} \\ + Q_{j_1}P_{j_2}Q_{j_4}P_{j_3} + Q_{j_2}P_{j_1}Q_{j_4}P_{j_3}. \quad (12)$$



FIG. 3: Configurations on the outside links not annihilated by J_{out}

Each H_j therefore mixes each of four sets of three configurations among themselves; one of these sets is illustrated in fig. 2, while the other three sets are given by changing the configurations on the outer links to one of the others in fig. 3. Because of the presence of J_{out} , H_j does not mix any net configurations with any non-net configurations. Using (10) then ensures that $H_j|\Psi\rangle = 0$ for j any set of eight links of the form illustrated in fig. 2, i.e. the four spins on a face and the four touching two adjacent vertices. There are thus four such terms for each face on the lattice. Since $|\Psi\rangle$ can be rewritten in terms of nets on the dual lattice as well, the analogous projectors $H_{\hat{j}}$ also annihilating $|\Psi\rangle$ can be defined by repeating the above arguments acting on the $|\hat{0}\rangle_l, |\hat{1}\rangle_l$ basis on the dual lattice. The full Hamiltonian is then

$$H = \sum_v H_v + \sum_f H_f + \varepsilon \left(\sum_j H_j + \sum_{\hat{j}} H_{\hat{j}} \right), \quad (13)$$

where ε is a coupling strength included for convenience.

Numerical results— We have confirmed by exact diagonalization that the Hamiltonian (13) has exactly four ground states when $\varepsilon > 0$, as required for doubled Fibonacci topological order. More importantly, we have also checked that H is gapped as for $\varepsilon > 0$. Our data is shown in figs. 4 and 5. In fig. 4, we show the gap Δ to the first excited state as a function of ε for different system sizes. To illustrate how the gap survives in the thermodynamic limit, we show in fig. 5 its finite-size scaling. This suggests that the gap is indeed non-zero, with an estimate of $\Delta \approx 0.025$ in the thermodynamic limit for $\varepsilon = 0.8$. We checked as well the gap including the

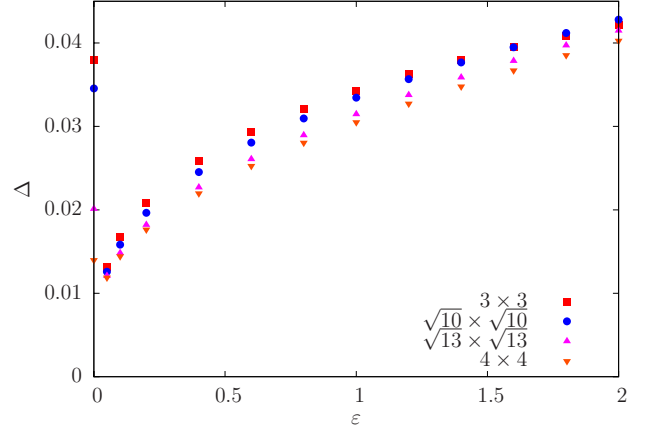


FIG. 4: The gap to the first excited state as a function ε for different system sizes.

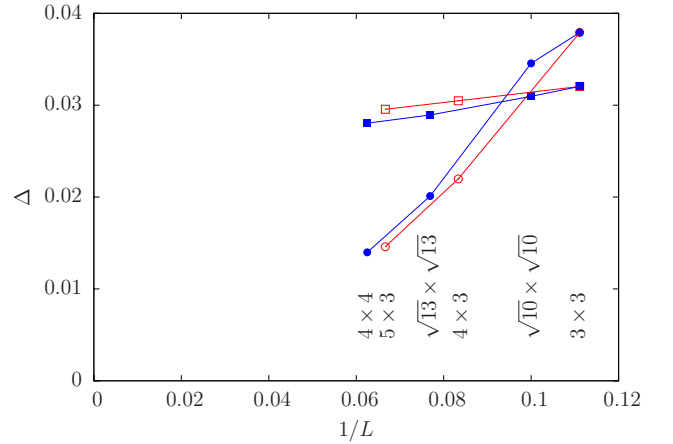


FIG. 5: The gap to the first excited state as a function of the inverse system size at $\varepsilon = 0$ (circles, no Jones-Wenzl terms) and at $\varepsilon = 0.8$ (squares). Lines guide the eye. The system is gapless at $\varepsilon = 0$ and gapped at $\varepsilon = 0.8$.

H_j but omitting the dual projectors $H_{\hat{j}}$, and obtained about half this value.

We diagonalized H by the Lanczos method, using a parallel and highly scalable exact diagonalization code [21]. The spins live on the links of a $L_1 \times L_2$ square lattice with periodic boundary conditions, so that the clusters contain $2L_1L_2$ sites. Clusters with periodic boundary conditions of size 3×3 , $\sqrt{10} \times \sqrt{10}$, 4×3 , $\sqrt{13} \times \sqrt{13}$, 5×3 , and 4×4 were diagonalized. The 3×3 , $\sqrt{10} \times \sqrt{10}$, $\sqrt{13} \times \sqrt{13}$, and 4×4 clusters have the symmetry of the infinite lattice. We exploit translation symmetry to reduce the Hilbert-space size, so the largest Hilbert space size is about 2.7×10^8 for the 4×4 lattice. It is not very large, but we have complicated multi-spin interactions and there are many matrix elements per row; e.g. roughly 16000 for the 4×4 lattice. The lowest eigenvalues for $\varepsilon > 0$ are in the zero-momentum sector.

Conclusion— We have presented strong evidence for a magnetic model with a gap and doubled Fibonacci topological order. Our results illustrate how to simplify the string-net model while still remaining in the same topological phase. We believe this is the “minimal” gapped Hamiltonian with an exact ground state in this phase, thus completing the program of Ref. 8 in the Fibonacci case. By expressing the ground state in terms of a topological quantity, the chromatic polynomial, our construction also illuminates precisely how topology appears in this ground state. Moreover, since H is the sum of non-commuting projectors, the model is not exactly solvable, so that the excitations are dynamical, and not just fixed defects.

It would be very interesting to understand how this Hamiltonian could be simplified further (e.g. by omitting J_{out} in H_j) while still preserving the same topological order. It would also be illuminating to apply a similar analysis to other models with non-abelian topological order. A good starting point may be the models of e.g. Refs. 22, 23, related to the one considered here. Another interesting topic for further study is the presumed quantum critical point at $\epsilon = 0$ describing a transition out of the topological phase; an analogous point also occurs in the correspondingly deformed toric code.

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